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New Upper Bound for the B-Spline Basis Condition Number

II. A Proof of de Boor's 2^k-Conjecture

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For the p-norm condition number $\kappa_{k,p}$ of the B-spline basis of order k we prove the upper estimate $\kappa_{k,p} < k2^k$. This proves de Boor's 2^k-conjecture up to a polynomial factor. © 1999 Academic Press

1. INTRODUCTION

It is of central importance for working with B-spline series that its condition number is bounded independently of the underlying knot sequence. This fact was proved by C. de Boor in 1968 for the sup-norm and in 1973 for any L_p-norm (see [B1] for references). In the paper [B2] he gave the direct estimate

$$\kappa_{k,p} < k9^k \tag{1.1}$$

for $\kappa_{k,p}$, the worst condition number with respect to the p-norm of a B-spline basis of order k, and conjectured that the real value of $\kappa_{k,p}$ grows like 2^k,

$$\kappa_{k,p} \sim 2^k, \tag{1.2}$$

which is seen to be far better than (1.1).

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The conjecture was based on numerical calculations of some related constants which moreover gave some evidence that the extreme case occurs for a knot sequence without interior knots (the so-called Bernstein knots). Maybe due to this reason, a few papers devoted to the 2^k-conjecture for $\kappa_{k,p}$ were concerned only with the "Bernstein knots" conjecture for the extreme knot sequence, see [B3, C, Ly, S].

These papers gave further support for de Boor's conjecture (1.2), in particular T. Lyche [Ly] obtained a lower bound for $\kappa_{k,\infty}$ from which it follows [S] that

$$\kappa_{k,p} > ck^{-1/p}2^k. \tag{1.3}$$

In the unpublished manuscript [SS1] we returned to de Boor's direct approach in [B2], and considered the possibility of improving his 9^k-estimate by several modifications of his method. In particular, a slight revision based on Kolmogorov's estimate for intermediate derivatives had shown that

$$\kappa_{k,p} < k\gamma^k, \qquad \gamma = 6.25.$$

In the previous paper [SS2] we developed a further approach to obtain

$$\kappa_{k,p} < k^{1/2}4^k.$$

In this paper using the same approach we give a surprisingly short and elementary proof of

THEOREM 1. *For all k and all $p \in [1, \infty]$,*

$$\kappa_{k,p} < k2^k. \tag{1.4}$$

With respect to (1.2)–(1.3), this confirms C. de Boor's conjecture up to a polynomial factor.

We show also that the optimal factor which can be obtained in (1.4) within this approach is $k^{1/2}$ and discuss further possible approaches by which this factor could be removed.

2. CONDITION NUMBER AND RELATED CONSTANTS

Let $\{\hat{N}_j\}$ be the B-spline basis of order k on a knot sequence $t = (t_j)$, $t_j < t_{j+k}$, normalized with respect to the L_p-norm $(1 \leqslant p \leqslant \infty)$, i.e.,

$$\hat{N}_j(x) = (k/(t_{j+k} - t_j))^{1/p}\, N_j(x),$$

where $\{N_j\}$ is the B-spline basis which forms a partition of unity. Recall here that

$$N_j(t) = ([t_j, ..., t_{j+k-1}] - [t_{j+1}, ..., t_{j+k}])(\cdot - t)_+^{k-1}$$

and that

$$N_j(x) > 0, \quad x \in (t_j, t_{j+k}); \qquad N_j(x) = 0, \quad x \notin [t_j, t_{j+k}]; \qquad \sum N_j = 1.$$

The condition number of the L_p-normalized basis $\{\hat{N}_j\}$ is defined as

$$\begin{aligned}\kappa_{k,p,t} &:= \sup_b \frac{\|b\|_{l_p}}{\|\sum b_j \hat{N}_j\|_{L_p}} \sup_b \frac{\|\sum b_j \hat{N}_j\|_{L_p}}{\|b\|_{l_p}} \\ &= \sup_b \frac{\|b\|_{l_p}}{\|\sum b_j \hat{N}_j\|_{L_p}},\end{aligned}$$

where the L_p-norm is taken with respect to the smallest interval containing the knot sequence (t_i).

The last equality in the above definition follows from normalization

$$\hat{N}_j(x) = M_j^{1/p}(x)\, N_j^{1/q}(x), \qquad M_j(x) := \frac{k}{t_{j+k} - t_j} N_j(x), \qquad \int M_j(x)\, dx = 1,$$

so that

$$\begin{aligned}\left\|\sum b_j \hat{N}_j\right\|_{L_p} &= \left\|\sum b_j M_j^{1/p} N_j^{1/q}\right\|_{L_p} \leqslant \left\|\left(\sum b_j^p M_j\right)^{1/p} \left(\sum N_j\right)^{1/q}\right\|_{L_p} \\ &= \left\|\left(\sum b_j^p M_j\right)^{1/p}\right\|_{L_p} = \left\|\sum b_j^p M_j\right)\right\|_{L_1}^{1/p} \\ &\leqslant \|b\|_{l_p},\end{aligned}$$

with equalities for $b_j = ((t_{j+k} - t_j)/k)^{1/p}$.

The worst B-spline condition number is defined then as

$$\kappa_{k,p} := \sup_t \kappa_{k,p,t}.$$

Its value gives a measure for the uniform stability of the B-spline basis and is important for numerical calculations with B-splines.

Following [B2] we introduce now related constants that are upper bouds for $\kappa_{k,p}$. This has been done already in [SS2] but for convenience of the reader we state here again the relevant lemmas. More details can be found in [B1, B2, S].

LEMMA A. *Let H_i be the class of functions $h \in L_q$ such that*

$$\text{(1)} \quad \operatorname{supp} h \subset [t_i, t_{i+k}]$$

$$\text{(2)} \quad \int hN_j = \delta_{ij}$$

and let

$$D_{k,p} := \sup_t \sup_i \inf_{h \in H_i} \{(t_{i+k} - t_i)^{1/p} \|h\|_q\},$$

where $1/p + 1/q = 1$. *Then*

$$\kappa_{k,p} \leqslant D_{k,p}.$$

Now set

$$\psi_i(x) := \frac{1}{(k-1)!} \prod_{\nu=1}^{k-1} (x - t_{i+\nu}).$$

Then an easy way for obtaining $h \in H_i$ is to set $h = (g\psi_i)^{(k)}$ for some appropriate smooth function g. We formulate this as

LEMMA B. *Let G_i be the class of functions g such that*

$$\text{(1)} \quad g\psi_i \in W_q^k[t_i, t_{i+k}],$$

$$\text{(2)} \quad g\psi_i = \begin{cases} 0, & k\text{-fold at } t_i, \\ \psi_i, & k\text{-fold at } t_{i+k}, \end{cases}$$

and let $G_i^{(k)} := \{(g\psi_i)^{(k)} : g \in G_i\}$. *Then*

$$G_i^{(k)} \subset H_i.$$

Combining Lemmas A and B gives

COROLLARY. $\kappa_{k,p} \leqslant B_{k,p} := \sup\limits_t \sup\limits_i \inf\limits_{g \in G_i} \{(t_{i+k} - t_i)^{1/p} \|(g\psi_i)^{(k)}\|_q\}.$

Finally, due to the local character of the quantity $B_{k,p}$, it is sufficient to restrict attention to the meshes Δ of the form

$$\Delta = (t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k), \qquad t_0 < t_k.$$

Set also

$$\omega(x) := \omega_\Delta(x) = \frac{1}{(k-1)!} \prod_{i=1}^{k-1} (x - t_i) = \psi_0(x), \tag{2.1}$$

and

$$N(t) = N_\Delta(t) = ([t_0, ..., t_{k-1}] - [t_1, ..., t_k])(\cdot - t)_+^{k-1}.$$

LEMMA C. *For ω given via Δ as in (2.1), let G_Δ be the class of functions g such that*

$$(1) \quad g\omega \in W_q^k[t_0, t_k],$$

$$(2) \quad g\omega = \begin{cases} 0, & k\text{-fold at } t_0, \\ \omega, & k\text{-fold at } t_k, \end{cases}$$

and let

$$B_{k,p} := \sup_{\Delta} \inf_{g \in G_\Delta} (t_k - t_0)^{1/p} \, \|(g\omega)^{(k)}\|_q.$$

Then

$$\kappa_{k,p} \leqslant B_{k,p} \leqslant B_{k,1}. \tag{2.2}$$

Remark. Lemma A is taken from [B2, p. 123] whereas Lemma B and, respectively, C are somewhat more accurate versions of what is given in [B2, Eq. (4.1)]. Namely, they show the possibility to choose a smoothing function g *depending* on ω. C. de Boor's estimate of $B_{k,1}$ resulting in (1.1) was based on the inequalities

$$B_{k,1} \leqslant \inf_{g \in G_\Delta} \sup_{\omega} \|(g\omega)^{(k)}\|_\infty \leqslant \inf_{g \in G_\Delta} \sum_{i=m}^{k} \binom{k}{m} \|g^{(m)}\|_\infty \sup_{\omega} \|\omega^{(k-m)}\|_\infty$$
$$\leqslant \sum_{i=m}^{k} \binom{k}{m} \|g_*^{(m)}\|_\infty \sup_{\omega} \|\omega^{(k-m)}\|_\infty,$$

with some special choice of $g_* \in G := \bigcap G_\Delta$ that is seen to be *independent* of ω. Notice, that in the latter sum for any choice of $g_* \in G$ the term with $m = k$ is equal at least to 4^{k-1} (see [B2, p. 132]).

3. PROOF OF THEOREM 1

The idea in the previous paper [SS2] was to choose $g \in G_\Delta$ as the indefinite integral of the L_∞-normalized B-spline, i.e.,

$$g_\Delta(x) := \frac{k}{t_k - t_0} \int_{t_0}^{x} N_\Delta(t)\, dt.$$

Then, the inclusion $g_\Delta \in G_\Delta$ is almost evident (see [SS2]), and thus we can majorize the constant $B_{k,1}$ by

$$B_{k,1} \leqslant S_{k,1} := \sup_\Delta\, (t_k - t_0)\, \|s_\Delta^{(k)}\|_\infty, \tag{3.1}$$

where

$$s_\Delta := g_\Delta \omega_\Delta. \tag{3.2}$$

Notice that $\operatorname{supp} s_\Delta^{(k)} \subset [t_0, t_k]$, so that actually the L_∞-norm in (3.1) is taken over $[t_0, t_k]$.

In view of

$$(t_k - t_0)\, s_\Delta^{(k)}(x) = k \sum_{m=1}^{k} \binom{k}{m} N_\Delta^{(m-1)}(x)\, \omega_\Delta^{(k-m)}(x), \tag{3.3}$$

we showed in [SS2] that, for any Δ and $m = 1, ..., k$,

$$\|N_\Delta^{(m-1)} \omega_\Delta^{(k-m)}\|_\infty \leqslant \binom{k-1}{m-1}, \tag{3.4}$$

which, by Lemma C and (3.1)–(3.3), implies the bound

$$\kappa_{k,p} < k^{1/2} 4^k.$$

Here we improve (3.4) by

LEMMA 1. *For any Δ, and $m = 1, ..., k$*

$$\|N_\Delta^{(m-1)} \omega_\Delta^{(k-m)}\|_\infty \leqslant 1. \tag{3.5}$$

Now, by (3.1)–(3.5) and Lemma C,

$$\kappa_{k,p} \leqslant S_{k,1} \leqslant k \sum_{m=1}^{k} \binom{k}{m} = k(2^k - 1) < k 2^k$$

which proves Theorem 1.

Remark. If ω has a multiple zero

$$\tau_\nu := t_{\mu_\nu} = t_{\mu_\nu+1} = \cdots = t_{\mu_\nu+p_\nu-1}$$

of multiplicity p_ν, then $N_\Delta^{(k-p_\nu)}$ has a jump at τ_ν. In this case we can define the value $N_\Delta^{(k-p_\nu+q)}(\tau_\nu)\,\omega^{(p_\nu-1-q)}(\tau_\nu)$ as a limit either from the left or from the right. This limit is equal to zero, if $\tau_\nu \in (t_0, t_k)$. Also this definition justifies the equality (3.3).

4. LEE'S FORMULA AND A LEMMA OF INTERPOLATION

For arbitrary $r \in \mathbf{Z}_+$ and $t \in \mathbf{R}$, set

$$\phi_r(x, t) := \frac{1}{r!}(x-t)_+^r,$$

and define $Q_{\delta_1}(x, t)$ and $Q_{\delta_2}(x, t)$ as algebraic polynomials of degree $k-1$ with respect to x that interpolate the function $\phi_{k-1}(\cdot, t)$ on the meshes

$$\delta_1 = (t_0, t_1, ..., t_{k-1}), \qquad \delta_2 = (t_1, ..., t_{k-1}, t_k),$$

respectively.

The following nice formula is due to Lee [L].

LEMMA D [L]. *For any Δ,*

$$N(t)\,\omega(x) = Q_{\delta_1}(x, t) - Q_{\delta_2}(x, t). \tag{4.1}$$

Proof [L]. The difference on the right-hand side is an algebraic polynomial of degree $k-1$ with respect to x that is equal to zero at $x = t_1, ..., t_{k-1}$, hence

$$Q_{\delta_1}(x, t) - Q_{\delta_2}(x, t) = c(t) \prod_{i=1}^{k-1} (x - t_i).$$

Further, since the leading coefficient of the Lagrange interpolant to f on the mesh $(\tau_i)_{i=1}^k$ is equal to $[\tau_1, ..., \tau_k]\,f$, we have

$$c(t) = ([t_0, ..., t_{k-1}] - [t_1, ..., t_k])\,\phi_{k-1}(\cdot, t) =: \frac{1}{(k-1)!}\,N(t),$$

and the lemma is proved.

We will use Lee's formula (4.1) to evaluate the product $N^{(m-1)}(t)\,\omega^{(k-m)}(t)$ by taking the corresponding partial derivatives with respect to x and t in (4.1) and setting $x=t$.

Our next two lemmas give a bound for the values obtained in that way on the right-hand side of (4.1).

For arbitrary $p\in\mathbf{N}$, $p>r$, and any sequence

$$\delta=(\tau_0\leqslant\tau_1\leqslant\cdots\leqslant\tau_p),$$

define, for a fixed t,

$$Q_t(x):=Q(x,t):=Q(x,t;\phi_r,\delta)$$

as the polynomial of degree p with respect to x that interpolates $\phi_r(\cdot,t)$ at δ.

LEMMA 2. *For any admissible p, r, t, δ,*

$$0\leqslant Q_t^{(r)}(x)|_{x=t}\leqslant 1, \tag{4.2}$$

where the derivative is taken with respect to x.

Proof. First we prove

(A) *The case $r=0$.* Then $Q_t(\cdot)$ is a polynomial of degree $\leqslant p$ that interpolates, for this fixed t, the function

$$(x-t)_+^0:=\begin{cases}1, & x\geqslant t;\\ 0, & x<t.\end{cases}$$

We have to prove that

$$0\leqslant Q_t(x)|_{x=t}\leqslant 1 \tag{4.3}$$

and distinguish the following cases:

(A1) If $t=\tau_i$ for some i, then (4.3) is evident.

(A2) If all the points of interpolation lie either to the left or to the right of t, i.e., if

$$\tau_p<t, \qquad \text{or} \qquad t<\tau_0,$$

then

$$Q_t\equiv 0, \qquad \text{or} \qquad Q_t\equiv 1,$$

respectively, and (4.3) holds.

(A3) If t lies between two points, i.e., for some v

$$\tau_0 \leqslant \cdots \leqslant \tau_v < t < \tau_{v+1} \leqslant \cdots \leqslant \tau_p,$$

then in view of $Q_t'(x) = [Q_t - \phi_0(\cdot, t)]'(x)$ for $x \neq t$, the polynomial $Q_t'(x)$ has at least v zeros on the left of τ_v, and at least $p - v - 1$ zeros on the right of τ_{v+1}, which gives $p - 1$ zeros in total. Hence Q_t' has no zeros in (τ_v, τ_{v+1}), so that Q_t is monotone in (τ_v, τ_{v+1}), that is,

$$0 = Q_t(\tau_v) < Q_t(t) < Q_t(\tau_{v+1}) = 1.$$

(B) *The case* $r > 0$. This case is reduced to the case $r = 0$ by Rolle's theorem. The difference $\phi_r(\cdot, t) - Q_t$ has $p + 1$ zeros (counting multiplicity), thus its rth derivative $\phi_0(\cdot, t) - Q_t^{(r)}$ must have at least $p + 1 - r$ changes of sign.

If (4.2) does not hold, then this function does not change sign at $x = t$, and $Q_t^{(r)}$ is a polynomials of degree $p - r$ that interpolates $\phi_0(\cdot, t)$ at $p - r + 1$ points all distinct from t. But according to the Case (A3) this would imply (4.2), a contradiction.

Hence, (4.2) holds, and the lemma is proved.

LEMMA 3. *For any admissible* p, r, t, δ,

$$0 \leqslant (-1)^s \frac{\partial^{r-s}}{\partial x^{r-s}} \frac{\partial^s}{\partial t^s} Q(x, t)|_{x=t} \leqslant 1. \tag{4.4}$$

Proof. Let l_i be the fundamental Lagrange polynomials of degree p for the mesh δ, i.e., $l_i(\tau_j) = \delta_{ij}$. Then $Q_t = Q(\cdot, t)$, which is the Lagrange interpolant to $\phi_r(\cdot, t)$, can be expressed as

$$Q(x, t) = \frac{1}{r!} \sum_{i=0}^{p} (\tau_i - t)_+^r \, l_i(x).$$

Thus, we obtain

$$(-1)^s \frac{\partial^s}{\partial t^s} Q(x, t) = \frac{1}{(r-s)!} \sum_{i=0}^{p} (\tau_i - t)_+^{r-s} \, l_i(x).$$

It is readily seen that

$$Q_{0,t}(x) := Q_0(x, t) := (-1)^s \frac{\partial^s}{\partial t^s} Q(x, t)$$

is a polynomial of degree p with respect to x that interpolates

$$\phi_{r-s}(\cdot, t) = \frac{1}{(r-s)!} (\cdot - t)_+^{r-s}$$

at the same mesh δ. Now (4.4) follows from Lemma 2.

5. PROOF OF LEMMA 1

We need to bound

$$N^{(s)}(t)\, \omega^{(k-1-s)}(t) = N^{(s)}(t)\, \omega^{(k-1-s)}(x)|_{x=t}, \qquad s = 0, 1, ..., k-1.$$

Now according to Lemma D

$$N^{(s)}(t)\, \omega^{(k-1-s)}(x) = \frac{\partial^{k-1-s}}{\partial x^{k-1-s}} \frac{\partial^s}{\partial t^s} Q_{\delta_1}(x, t) - \frac{\partial^{k-1-s}}{\partial x^{k-1-s}} \frac{\partial^s}{\partial t^s} Q_{\delta_2}(x, t),$$

and by Lemma 3 for any δ

$$0 \leqslant (-1)^s \frac{\partial^{k-1-s}}{\partial x^{k-1-s}} \frac{\partial^s}{\partial t^s} Q_\delta(x, t)|_{x=t} \leqslant 1.$$

Hence, since both terms are of the same sign and of absolute value $\leqslant 1$,

$$|N^{(s)}(t) \cdot \omega^{(k-1-s)}(t)| \leqslant 1,$$

which proves Lemma 1.

6. ON THE FACTOR k IN THEOREM 1

Numerical computations [B3] show that

$$\kappa_{k,p} \leqslant c 2^k, \tag{6.1}$$

so a natural question is whether the factor k in the bound

$$\kappa_{k,p} < k 2^k \tag{6.2}$$

of Theorem 1 can be removed.

A simple example will show now that within the particular method we used in Section 3 (see (3.1)), an extra polynomial factor $\sqrt{k}$ appears unavoidably. Namely, one can prove that for some choice of Δ_*

$$S_{k,1} \geqslant (t_k - t_0)\, \|s_{\Delta_*}^{(k)}\|_\infty \geqslant ck^{1/2}2^k.$$

In fact, in the case of the Bernstein knots Δ_ν in $[0, 1]$, i.e., for

$$\omega_\nu(x) = \frac{1}{(k-1)!}\, x^\nu (x-1)^{k-1-\nu},$$

we have

$$N_\nu(x) = \binom{k-1}{\nu} x^{k-1-\nu}(1-x)^\nu,$$

and obtain

$$s_{\Delta_\nu}^{(k)}(x) = \frac{k}{(k-1)!}\binom{k-1}{\nu} \times \sum_{m=1}^{k} \binom{k}{m} [x^{k-1-\nu}(1-x)^\nu]^{(m-1)}\, [x^\nu (x-1)^{k-1-\nu}]^{(k-m)}.$$

It is not hard to see that at $x = 1$ the mth term vanishes, unless $m = \nu + 1$, which gives

$$|s_{\Delta_\nu}^{(k)}(1)| = \frac{k}{(k-1)!}\binom{k-1}{\nu} \cdot \binom{k}{\nu+1}\, \nu!\,(k-1-\nu)! = k\binom{k}{\nu+1}.$$

With this, we take $\nu_* + 1 = \lfloor k/2 \rfloor$ to obtain

$$|s_{\Delta_*}(1)| = k\binom{k}{\lfloor k/2 \rfloor} > ck^{1/2}2^k.$$

7. POSSIBLE REFINEMENTS

We describe here some further approaches that may permit removal of the polynomial factor in the upper bound for the sup-norm condition number $\kappa_{k,\infty}$.

(1) The first approach is to majorize $\kappa_{k,\infty}$ using the intermediate estimate (2.2) with the value $B_{k,\infty}$ instead of $B_{k,1}$ used in Theorem 1, that is,

$$\kappa_{k,\infty} \leqslant B_{k,\infty}.$$

Then the desired 2^k-bound without an extra factor will follow from the following

Conjecture. For any $\omega = \omega_\Delta$, there exists a function $g_* \in G_\Delta$ such that

$$\operatorname{sign} g_*^{(m)}(x) = \operatorname{sign} \omega^{(k-m)}(x), \qquad x \in [t_0, t_k], \quad m = 1, ..., k. \tag{7.1}$$

This conjecture implies that

$$\| g_*^{(m)} \omega^{(k-m)} \|_{L_1[t_0, t_k]} = \left| \int_{t_0}^{t_k} g_*^{(m)}(x)\, \omega^{(k-m)}(x)\, dx \right|.$$

Then observe that, because of the boundary conditions satisfied by g_* and the way g_* and ω_Δ are normalized,

$$(-1)^m \int_{t_0}^{t_k} g_*^{(m)}(x)\, \omega^{(k-m)}(x)\, dx = \int_{t_0}^{t_k} g'_*(x)\, \omega^{(k-1)}(x)\, dx = 1.$$

Hence

$$\| g_*^{(m)} \omega^{(k-m)} \|_{L_1[t_0, t_k]} = 1, \qquad m = 1, ..., k, \tag{7.2}$$

and using this bound, one could show, exactly as in Section 3, that

$$\kappa_{k,\infty} \leqslant B_{k,\infty} \leqslant \sum_{m=1}^{k} \binom{k}{m} = 2^k - 1.$$

Remark. (1) A function g_* satisfying (7.1) should in a sense be close to the function g_Δ considered in Section 3 (though it is not necessarily unique). Moreover, g_Δ can serve as g_* for the polynomials ω_{Δ_ν} with the Bernstein knots

$$\omega_{\Delta_\nu}(x) = c(x - t_0)^\nu\, (x - t_k)^{k-1-\nu}.$$

Also, it looks quite probable that, even though the equality (7.2) is not valid with $g_* = g_\Delta$ for arbitrary Δ, there holds

$$\| g_\Delta^{(m)} \omega_\Delta^{(k-m)} \|_{L_1[t_0, t_k]} \leqslant c, \qquad m = 1, ..., k,$$

that is, for the B-spline $M_\Delta(x) = (k/(t_k - t_0))\, N_\Delta(x)$ we have

$$\|M_\Delta^{(m-1)} \omega_\Delta^{(k-m)}\|_{L_1[t_0, t_k]} \leqslant c.$$

(2) Another possibility to improve the result of Theorem 1 would be to find a sharp bound for one of the related constants considered in [S]. In this respect it is known, e.g., that

$$\kappa_{k,\infty} \leqslant E_{k,p}^{-1}, \tag{7.3}$$

where

$$E_{k,p} := \inf_\Delta \inf_j \inf_{c_i} \left\{ \left\| N_j - \sum_{i \neq j} c_i N_i \right\|_p \right\}.$$

In particular, there is equality in (7.3) for $p = \infty$.

The hope is to prove that the knot sequence at which the value $E_{k,p}$ is attained for $p = 1$ or $p = 2$ is the Bernstein one, in which case the inequalities

$$E_{k,1}^{-1} < c2^k, \qquad \text{or} \qquad E_{k,2}^{-1} < c2^k$$

would follow. (It is known that the Bernstein knot sequence is not extreme for $p = \infty$, see [B3].)

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